7 Problems with ordinary differential equations

We have now determined how to solve a system of algebraic equations; that is, equations that depend on *n* variables and we found solutions that satisfied those *n* equations. We will now consider solve systems of analytic equations. We will start, however, by describing a single analytic equation.

An analytic equation is an equation involving a function and derivatives or integrals of that function. In general, however, we will restrict ourselves to derivatives, as we can always reformulate an integral equation to an equation with derivatives, or a *differential equation*.

Classification of ordinary differential equations

Given a function *y* of one variable *t*, a first-order ordinary differential equation (ODE) is any equation involving the function, the variable, and the derivative of the function with respect to the variable; for example,

$$\left(2t + \frac{\mathrm{d}}{\mathrm{d}t} y(t)\right)^2 - y^2(t)\frac{\mathrm{d}}{\mathrm{d}t} y(t) + \cos(t y(t)) + 1 = 0 .$$

In general, a first-order ODE may be written as

$$F\left(t, y(t), \frac{\mathrm{d}}{\mathrm{d}t} y(t)\right) = 0 .$$

If the ODE is independent of *t*, it is called *time-independent* and may be written as

$$F\left(y(t), \frac{\mathrm{d}}{\mathrm{d}t} y(t)\right) = 0 .$$

For example, the following are time-independent first-order ODEs:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} y(t)\right)^2 - y^2(t)\frac{\mathrm{d}}{\mathrm{d}t} y(t) + \cos(y(t)) + 1 = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) y^2(t) - 1 = 0$$

The only instances of the variable *t* are as the argument to the function *y*.

An ODE is called *linear* (LODE) if it is a linear combination of the basis functions

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t), y(t), 1$$

where the coefficients can be functions of *t*. For example, the following are both LODEs:

$$t\frac{\mathrm{d}}{\mathrm{d}t} y(t) + 3y(t) + \sin(t) = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} y(t) - (t^2 - 1) y(t) - 5t = 0 .$$

The coefficient of 1 is often referred to as the *forcing function* and is sometimes written on the other side:

$$t\frac{\mathrm{d}}{\mathrm{d}t}y(t)+3y(t)=-\sin(t) \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}y(t)-\left(t^2-1\right)y(t)=5t\,.$$

If the coefficient of 1 is zero, it is said to be a *homogeneous* ODE (HODE as the *linear* is implied), so homogeneous variations of the above two linear ordinary differential equations are

$$t\frac{\mathrm{d}}{\mathrm{d}t}y(t)+3y(t)=0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}y(t)-\left(t^2-1\right)y(t)=0\,,$$

respectively.

If the coefficients of $\frac{\mathrm{d}}{\mathrm{d}t}y(t)$ and $y(t)$ do not depend on *t*, then it is a LODE with constant coefficients:

$$5\frac{\mathrm{d}}{\mathrm{d}t}y(t)+3y(t)=g(t) \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}y(t)-6y(t)=g(t)$$

where again $g(t)$ is the *forcing function*. If the forcing function is a constant, it is a time-independent LODE with constant coefficients, so for example:

$$5\frac{\mathrm{d}}{\mathrm{d}t}y(t)+3y(t)=1 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}y(t)-6y(t)=2\,,$$

and if the coefficient of 1 is zero, it is a time-independent HODE with constant coefficients:

$$5\frac{\mathrm{d}}{\mathrm{d}t}y(t)+3y(t)=0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}y(t)-6y(t)=0\,.$$

Implicit function theorem

Now, one very important theorem that you must understand: the implicit function theorem.

Given an equation $F\left(t,y(t),\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right)=0$, it is almost always possible to write this local in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)=f\left(t,y(t)\right)$$

for some new function *f* that depends on the original function *F*. Consequently, we will often focus on finding solutions to ODEs in this form.

We will look at many different types of problems: approximating a solution to

1. a first-order initial-value problem,
2. a system of first-order initial-value problems,
3. higher-order initial-value problems,
4. a system of higher-order initial-value problems,
5. a second-order boundary value problem.

We will see that the techniques used in Problem 1 trivialize to solving Problem 2, that Problems 3 and 4 can be reformulated as a system of first-order initial-value problems, and thus the solution can be found by the same technique we used in Problem 2, and for a boundary value problem, we will use

1. the *shooting method*, which uses the solution to Problem 1 together with a root-finding technique, and
2. a divided-difference method that uses the centred divided-difference rules we have previously used for approximating solutions to the derivative.

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